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INVERSE PROBLEMS IN SOLID MECHANICS AND RELEVANT FIELDS

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1. INTRODUCTION

The role of inverse problems in the areas of science and engineering is getting more important [1-5]. The inverse problems may be regarded as the problems concerning the determination of inputs or sources from outputs or responses, in contrast to direct problems in which outputs or responses are sought from inputs or sources. We can find many inverse problems in solid mechanics and relevant research areas.

In this paper classification of inverse problems arising in analyses of variation of physical quantities are made, and it will be stated that there are five kinds of inverse problems. Examples of these inverse problems and their treatments will be demonstrated with special emphasis on inverse problems in solid mechanics and relevant fields.

2. DEFINITION OF INVERSE PROBLEMS

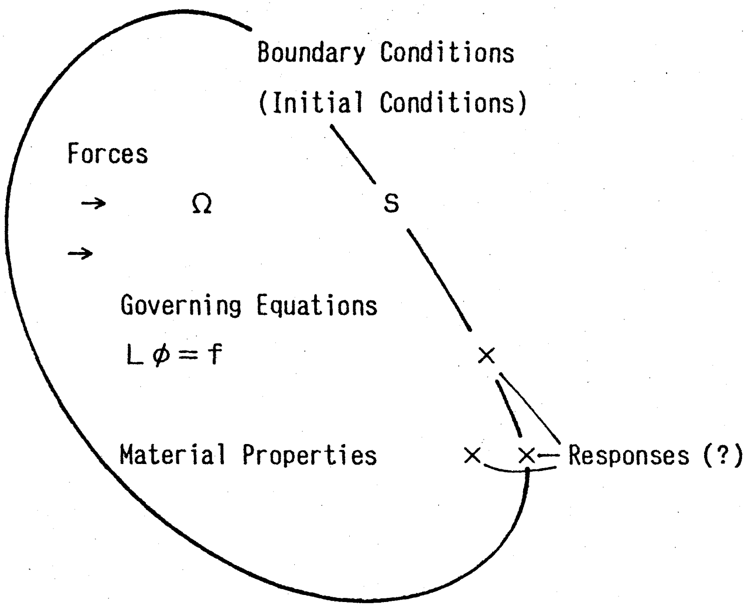
The term "inverse problems" is used in many ways, depending on research areas and sometimes on researchers. A rational definition of inverse problems can be made by referring to direct problems, which can be considered to be opposite to the inverse problems [3]. Let us consider an analysis of distribution of a field quantity ϕ representing physical states of concern. The governing equation is expressed as

$$L\phi = f \tag{1}$$

where L denotes an operator and f being a force acting in domain Ω. For a direct field analysis to be conducted, the information of the following items is indispensable, as is shown in Fig. 1.

(a) Domain Ω of concern and its boundary S.
(b) Governing equations representing the variation of the field quantity ϕ.
(c) Boundary conditions on the entire boundary of domain Ω, and initial conditions, if necessary.
(d) Forces f acting in domain Ω.
(e) Material properties involved in the governing equations.

When full information of these items is available in advance, outputs or responses can be uniquely determined. We can calculate the outputs or responses by using conventional numerical schemes, such as the finite



(a) Direct problems.

(a) Domain Ω, Boundary S
(b) Governing Equations
(c) Boundary Conditions →→→→→ Responses
(Initial Conditions) (Unknown Boundary Values on S)
(d) Forces (Quantity φ in Ω)
(e) Material Properties

(b) A direct analysis.

Fig. 1 Direct problems and direct analyses.

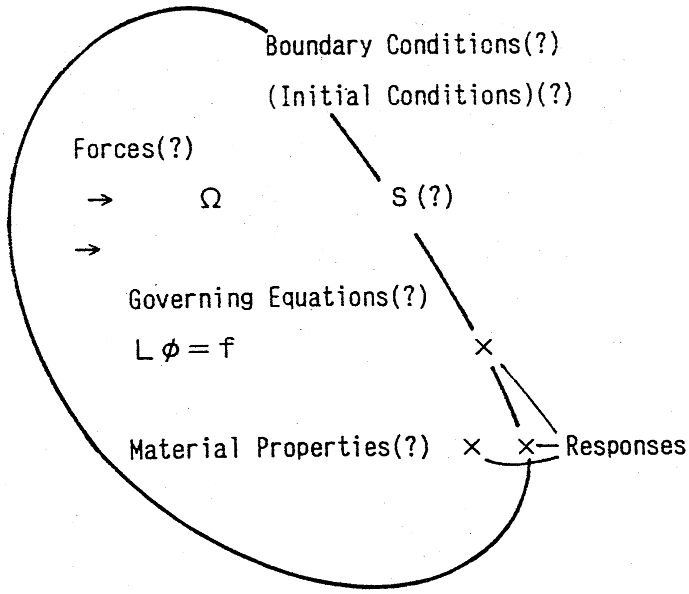
element method, the boundary element method, and the finite difference method.

If any of the requisites (a) to (e) is lacking, we can not calculate the distribution of field quantity ϕ. Those problems, which can not be classified as direct problems in the sense mentioned above, can be classified into the inverse problems.

For the field problem, there may be the following inverse problems corresponding to the lack of requisites (a) to (e) for direct analyses.

(a)' Determination of domain Ω, its boundary S or unknown inner boundary (domain/boundary inverse problems).

(b)' Inference of the governing equations (governing equation inverse problems).



(a) Inverse problems.

(a)' Domain Ω
(b) Governing Equations
(c) Boundary Conditions (Initial Conditions) →→→→→ (A) Unknown Boundary
↑ Responses
(d) Forces
(e) Material Properties

(b) An inverse analysis.

Fig. 2 Inverse problems and inverse analyses.

(c)' Estimation of the boundary conditions on the entire or partial boundary. Estimation of the initial conditions (initial condition/boundary condition inverse problems).
(d)' Determination of the forces f acting in Ω (force inverse problems).
(e)' Determination of the material properties defined in Ω and involved in the governing equations (material properties inverse problems).

Any combination of these inverse problems can be another inverse problem.

The inverse problems are inherently lacking in information as compared with that for direct problems. Additional information is necessary to conduct inverse analyses. As is shown in Fig. 2, outputs or responses can be used as additional information to conduct inverse analyses, in contrast to the direct problem, in which outputs or responses are determined. The information concerning outputs or responses can be obtained by measurements. Subsidiary information expressing physical a priori information and requirements may be used to achieve an effective estimation. In the followings examples of each of the five categories of inverse problems will be given.

3. DOMAIN/BOUNDARY INVERSE PROBLEMS

3.1 Electric Potential CT (Computed Tomography) Method

Electric potential method is used for monitoring crack length [7-18]. This method is based on the fact that the existence of cracks gives rise to disturbance in electric potential distributions. The location, size and shape of a two- or three-dimensional crack may be identified, if the measured distribution of electric potential is available. This crack identification from electric potential distributions can be recognized as one of the inverse problems of category (a)'. By introducing inverse analysis schemes formulated on the basis of the boundary element method, the present authors proposed the electric potential CT (computed tomography) method to identify the crack location, size and shape [5, 19].

3.2 Boundary Conditions of Crack Identification

In a usual direct boundary value problem of electrostatics, boundary S of domain Ω consists of boundary S_1 where the Dirichlet boundary condition is imposed, and boundary S_2 where the Neumann boundary condition is imposed. The values of the electric potential ϕ are prescribed on the Dirichlet boundary S_1, while the flux q is prescribed on the Neumann boundary S_2. Since thelocation, size and shape of the cracks are unknown in advance, the cracks constitute themselves other flux-free boundaries. BOundary S_0 denotes a hypothetical boundary called an "incompletely-prescribed boundary", which contains the cracks to be detected. None of the potential ϕ, the flux q and their combination can be given on S_0. The location of S_0 is not known in many cases. This gives rise to the incompleteness of the boundary conditions.

The existence of this incompletely-prescribed boundary induces a lack of boundary conditions, which makes the problem unsolvable without giving additional information. If we introduce supplementally an "over-prescribed boundary" S_3, where both ϕ and q are given by measuring the value of ϕ on some parts of the Neumann boundary S_2, the inverse problem of crack identification may be solved.

3.3 Inversion Analysis Schemes for Electric Potential CT Method

Two inverse analysis schemes, i.e. the inverse boundary integral equation method and the least residual method, were proposed based on the boundary element formulation. The former is formulated by referring to the formulation of the boundary element method.

The variation of the distribution of D.C. electric potential ϕ is determined using Laplace's equation. The value of ϕ at a point located on the boundary S is expressed by boundary integral, which involves values of potential ϕ, flux q and fundamental solution on boundary S. If the entire boundary S is divided into boundary elements, and nodes are introduced in these elements, the boundary integral equation can be written in the form of matrix equation, which relates potential and flux on boundary S. This equation can be solved for boundary values on the incompletely-prescribed boundary S_0, which contains cracks to be detected. The cracked portions in the plane S_0 are identified as flux-free portions in S_0. Thus, in the inverse boundary integral equation method the problem of crack identification of category (a)' is reduced to the problem of identification of boundary values, which is one of the inverse problems of category (c)'.

The other inverse scheme, i.e. least residual method, is based on boundary element potential calculations for assumed cracks. Cracks are assumed, which are expressed by various combinations of the plane S_0 containing cracks, crack location, size and shape. Then S_0 is separated into a cracked portion and the remaining uncracked portion. If the boundary values of q only are used on the over-prescribed boundary S_3, a direct analysis can be made, which gives the value of ϕ on the over-prescribed boundary S_3. To determine the most plausible crack, the square sum R of residuals is evaluated between the computed potential values $\phi^{(C)}$ and the measured values $\phi^{(M)}$ on S_3 using a weighting factor w. The most plausible crack is identified as the crack giving the smallest R value. Thus a quasi-solution is sought in the least residual method.

The present authors discussed the uniqueness of the inverse solution in the crack identification by the electric potential CT method [19, 20]. This discussion is based on the uniqueness of boundary value inverse problems with over-prescribed boundaries. It is found that cracks can be uniquely identified from the electric potential distribution, when the plane S_0 containing cracks is known in advance. When S_0 is not known, the electric potential distributions under two current application conditions are necessary to determine a single two-dimensional crack embedded in a body. To determine a single three-dimensional crack in an unknown plane, the electric potential distributions under three current application conditions are needed.

3.4 Simulations and Experiments of Crack Identification

Numerical simulations of the crack identification by the inverse boundary integral equation method were made for two-dimensional edge cracks, embedded cracks and plural embedded cracks [5]. This method was also applied to the identification of three-dimensional surface cracks and embedded cracks [21], and it was shown that the inverse boundary integral equation method was applicable to the determination of crack sizes and shapes.

The inverse boundary integral equation method, however, is sensitive to the errors involved in the potential data used in the inverse analyses. This is due to the ill-posedness of the boundary value inverse problems. Some constraints or regularizations are needed to obtain a reansonable estimate by this method.

The applicability of the least residual method for determining the crack location, size and shape of two- and three-dimensional cracks was demonstrated by computer simulations and experiments [22-27].

As was described in the foregoing, this crack lying in an unknown plane can not be uniquely identified from the potential distribution under only one current application condition. To ensure the uniqueness of the identification, multiple current application method was proposed, in which potential data measured under several current application conditions were processed simultaneously [23]. The experiments demonstrated the usefulness of the multiple current application method. To accomplish efficient identification of the crack by the least residual method, a hierarchical analysis scheme was adopted, in which a gradual refinement of assumed cracks were made.

Experiments were made for determining a three-dimensional surface crack in a steel plate by using the least residual method [25]. The height of the plane containing cracks and the crack shape in the plane were determined from the electric potential distributions.

For efficient identification of the three-dimensional surface crack by the least residual method, a hierarchical inverse analysis scheme was proposed, in which two-dimensional scanning inverse analyses were combined with full three-dimensional inverse analyses. By conducting this two-dimensional inverse analyses for many cross sections, a rough estimate of the height of the plane containing cracks and the crack shape was obtained. Following the two-dimensional analyses, three-dimensional least residual calculations were made to obtain more accurate estimation. Gradual refinement of assumed cracks was also incorporated in the hierarchical analyses.

It was shown that the height of the plane containing cracks and the crack shape can be estimated with good accuracy by the least residual method, even when the potential data are given on the back surface of the crack only.

This least residual inverse analysis scheme was applied to the identification of three-dimensional internal cracks embedded in a body [27]. Numerical simulations and experiments showed the usefulness of the method for identifying three-dimensional internal cracks.

To achieve a high speed computation on a workstation, an analytical expression of electric potential distribution reported by Johnson is used in the two-dimensional scanning analysis, and data base of electric potential distribution on a three-dimensional cracked body is utilized in the three-dimensional inverse analysis [28, 29]. This scheme can be applied with minor modification to identify a crack in a pipe or a plate with curvature. Numerical simulations and experiments has shown that the proposed scheme is useful for identifying cracks.

The crack identification by the least residual method can be treated as an optimization problem, when the residual R is used as an objective function [30, 31]. Several schemes based on the optimization method were proposed. Applicability of the scheme was assessed by numerical simulations and experiments. The conjugate gradient method and the modified Powell method were utilized as optimization schmes. A hierarchical scheme was introduced, in which the variable to be optimized were varied by referring to the results of previous analyses. It was found that the modified Powell method with the hierarchical procedure was effective for identification of the surface crack.

4. GOVERNING EQUATION INVERSE PROBLEMS

Governing equation inverse problems deal with the inference of a differential equation or equations governing the variation of field quantities of the present concern from observations of the field quantities

[32, 33]. The estimation of the governing equation can be reduced to the estimation of the order of the differential equation and coefficients involved in the equation. The present authors proposed the local derivatives method, in which the derivatives are determined approximately from observed field quantities using finite difference approximation and the sets of these derivative values are simultaneously used to identify the coefficients involved in the linear differential equation. This method can be applied to the estimation of nonlinear differential equation.

The local derivatives method was applied to the estimation of second order ordinary differential equation. Numerical simulations were made on the estimation of the governing equation from observed quantities at some points for certain distribution of the field quantity. When the order of the differential equation was known in advance, good estimation of the coefficients could be made from estimated derivative values.

When the order of the differential equation was not known in advance, the order has to be assumed. When the assumed order was lower than the real one, inconsistent estimation resulted for different sets of observed values. When the assumed order was coincident with the real one, consistent estimation could be made for different sets of the observed data. When the assumed order was higher than the real one, inconsistent results were obtained again. Then, the principle of parsimony works well, in which the assumed order employed in the estimation was increased gradually and the lowest order giving consistent estimation was adopted as the solution.

When the assumed order of the differential equation washigher than the real one, the estimated governing equation can be written as a linear combination of the original governing equation and its differentiated forms. The successive elimination method was then proposed, in which highest order term of the estimated governing equations was eliminated. Successive elimination finally yields consistent equations, which were coincident with the real governing equation. Numerical simulations of the successive elimination method were made and showed the usefulness of the elimination method.

A method was also proposed to reduce the effect errors involved in the observations. It was found that it was possible to estimate the governing equation from noisy observations.

5. INITIAL CONDITION/BOUNDARY CONDITION INVERSE PROBLEMS

Boundary value inverse problems deal with the estimation of unknown boundary conditions on incompletely prescribed boundaries using over-prescribed boundary value on other boundaries or in the domain. This kind of inverse problems are usually ill-conditioned.

A regularizing scheme using a priori information on the unknown variables was proposed for solving the ill-conditioned boundary value inverse problem [34]. The scheme was based on a multivariable constrained optimization algorithm for determining the most plausible solution satisfying inequality constraints deduced from information available in advance. To demonstrate the applicability of the scheme, it was applied to a boundary value inverse problem with truss-like structures. The nonpositiveness and nonnegativeness of unknown variables were used as the constraint. It was found that the scheme using the constraint was effective in obtaining reasonable estimates of the unknown boundary values and was rather insensitive to error in input data, while the unconstrained scheme was not.

A finite element-based inverse scheme was proposed for boundary value inverse problems, in which no boundary values were known in advance on the incompletely prescribed boundary [35, 36]. This scheme can be applied to the estimation of force and displacement on unaccessible boundary, such as contact region. This boundary value inverse problem is also ill-posed. For the regularization of the problem, function expansion method reducing the number of parameters was introduced. The use of power functions in the expansion was studied. For determining the optimum number of terms in the expansion, estimated error criterion was proposed and examined. The usefulness of the proposed regularization based on the function expansion and the estimated error criterion was shown using numerical simulations of contact force determination.

The Tikhonov regularization method was also introduced in the finite element inverse analysis scheme for regularizing the inverse solution. As stabilizing functionals, norm of forces and displacements, their derivatives, and their second derivatives were employed [36]. For determining the optimum value of the regularizing parameter in the Tikhonov regularization, the residual between the estimated and measured displacements on the over-prescribed boundary was used. Numerical simulations of estimation of contact force and displacement demonstrated the usefulness of the Tikhonov regularization method.

6. FORCE INVERSE PROBLEMS

A deterministic approach was proposed for identifying the force term in the governing equation [37, 38]. For steady-state heat conduction problems, it was shown that a volume integral of the intensity of a heat source multiplied by a harmonic function can be expressed by a boundary integral. This enables us to determine the intensity and the location of the heat source by combining the values of the boundary integrals evaluated using several harmonic functions.

Similar method was also developed for determining the force term governed by linear partial differential equation. The usefulness of the proposed method was demonstrated by numerical simulatios of heat source identification and concentrated body forces.

This method was further generalized for determining force terms in other linear differential governing equations from boundary observations. By introducing adjoint operator and subsidiary function, it is possible to estimate force terms involved in governing equation expressed in the form of differential equation.

A method was presented for determining initial residual stress fields from redistributed residual stresses measured at several points [39, 40]. Determination of the initial residual stress distribution was effectively achieved by introducing fundamental residual stress distribution functions, which satisfied physical requirements for the residual stress. The least residual criterion was used in the determination. An inverse sensitivity matrix was evaluated to estimate the effect of errors involved in the measurements and to select the best combination of measuring points. Numerical simulations of the determination of initial residual stress fields and the prediction of fatigue crack propagation lives showed the usefulness of the proposed method.

7. MATERIAL PROPERTIES INVERSE PROBLEMS

Material properties of individual components of a discrete system are estimated using the response of the system to several sets of external inputs [41]. Two inversion scheme were proposed for estimating the material properties: [K] matrix method and {C} vector method. The [K] matrix method determined a stiffness matrix of the system for estimating material properties. The {C} vector method was based on the stiffness equation expressed in terms of the {C} vector, whose components consist of parameters of material properties. The usefulness of these methods were shown by numerical simulations. These methods were successfully extended to the determination of material properties in a continuum by employing finite element discretization.

7. CONCLUSIONS

A brief review of inverse problems was made. Inverse problems were classified into domain/boundary inverse problems, governing equation inverse problems, boundary value/initial value inverse problems, force inverse problems, and material properties inverse problems. Examples of these inverse problems and inverse analysis schemes for them were demonstrated.

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